the present method gives a good approximation to the linear vibration theory.

#### Cable Network

Displacements and cable tensions of a statically determinant cable network shown in Fig. 3 are calculated by the present method and a linear finite element method (FEM) and results are compared. The axial stiffness of the cables is 785.5 kgf. A force of 1 kgf is imposed on node D from node E, which moves along the line of $X = Y \geq 0$, $Z = -1000$ mm. Displacements of node D and cable tensions corresponding to the node E position are shown in Table 1. When all cables are in tension ($X = Y \leq 500$), the results of the present method and FEM coincide. But when there is a slack cable ($X = Y > 500$), the model loses the statically determinant condition, and the linear FEM analysis results in a physically meaningless solution; cables 2 and 3 are subjected to compressive forces. Even in this case, the present method yields a mechanically balanced configuration of the network, in which cable 1 has a tension of 1.0 kgf, equivalent to the magnitude of the applied force, and is directed toward node E.

### Conclusion

This Note outlines a numerical analysis method applicable to general cable structures and presents three sample calculations. The present method can be applied to analyses of cable networks that may cause cable slackening; mechanical deformations by means of taking node positions, not displacements, as variables; considering full-order strains of cables; and employing a nonlinear cable stiffness.

### References

[1]Miura, K., Sakamaki, M., Yasaka, T., Ohtomo, I., Mitsugi, J., Yahagi, H., and Okazaki, K., "Research and Development of the Tension Truss Antenna," *Preprint 38th Congress of the International Astronautical Federation*, IAF-87-317, 1987.

[2]Belvin, W. K., Edighoffer, H. H., and Herstrom, C. L., "Quasi-Static Shape Adjustment of a 15 Meter Diameter Space Antenna," *Proceedings of AIAA/ASME/ASCE/AHS 28th SDM Conference*, Part 1, AIAA Paper 87-0869, April 1987, pp. 705–713.

[3]Herbert, J. J. and Bachtell, E. E., "Comparison of Tension Stabilized Structures for Large Space Antenna Reflectors," *Proceedings of 27th AIAA/ASME/ASCE/AHS Structures, Structural Dynamics and Materials Conference*, AIAA Paper 86-0876, May 1986, pp. 752–756.

# Closed Form Solution for Minimum Weight Design with a Frequency Constraint

B. P. Wang*
*University of Texas at Arlington,*
*Arlington, Texas 76019*

### Introduction

THE minimum weight design of structures with natural frequency constraints has been an active topic of research since the earlier work of Turner.[1] Recently, this problem has been re-examined by several researchers using approaches that combine finite-element analysis and numerical optimization techniques (see, e.g., Refs. 2–6). In this Note, closed form minimum weight design formulas are derived for one-dimensional discrete vibration systems (i.e., for systems with one degree of freedom per node) with one end fixed and the other end free. In practical application, shear building and torsional vibration of shafts can be represented as first approximation by such models. Our objective is to provide a closed form solution for optimum preliminary sizing for a specified natural frequency. Additionally, the results of the present Note can be used to generate test cases to verify the frequency constraint capability of general-purpose finite-element codes. Our results are derived based on the Lagrange multiplier techniques. It is very interesting to note that the results show that the optimal mode shape corresponding to the optimal design depends only on the specified natural frequency and the coefficients of the objective function. Using the optimal mode shape and the specified eigenvalue, the design parameter can then be computed from the eigenvalue problem. In the following sections, we will first state the design problems, and the main results will then be summarized in theorems. A numerical example is included to illustrate the results.

### Problem Statements

Consider the system shown in Fig. 1. The problem is to find minimum weight design for specified fundamental frequency of the system. This is problem A in which we ignore the spring mass. Mathematically, the problem can be stated as the following: Find spring constants $x_1$ to $x_n$ to minimize

$$W = \sum_{i=1}^{n} c_i x_i$$

subject to the constraint $\lambda_1 = p$ where $c_i$ are prescribed constants, $\lambda_1$ is the fundamental eigenvalue, and $p$ is a prescribed positive value. If the effect of structural mass is included, the optimal problem can be stated the same as the preceding if we define $c_i = \rho_i L_i^2 / E_i$ and $x_i = E_i A_i / L_i$ where $\rho_i$, $L_i$, $E_i$ are mass density, length, and Young's modulus, respectively, for element $i$ (see Fig. 2). This is referred to as problem B in the Note.

In these problems, the eigenvalue $\lambda_1$ is an implicit function of the design variables. They are related through the eigenvalue problem

$$[K]\{\phi\} = \lambda_1 [M]\{\phi\} \tag{1}$$

where $\lambda_1$ is the fundamental eigenvalue, $\{\phi\}$ the corresponding mode shape, $[K]$ and $[M]$ the system stiffness and mass matrices, respectively, and where

$$[K] = [K(x)] \tag{2}$$

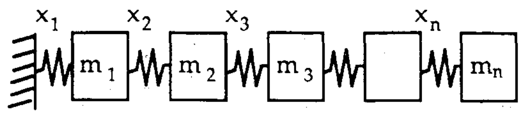


**Fig. 1 The *N*-degree-of-freedom spring mass system.**

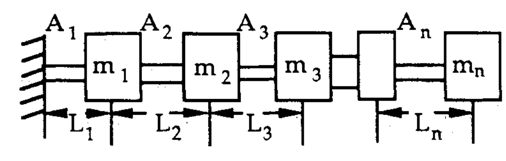


**Fig. 2 The *N*-degree-of-freedom axial vibration system.**

Received Aug. 23, 1988; revision received Nov. 13, 1989; accepted for publication Jan. 4, 1990. 

*Associate Professor, Department of Mechanical Engineering. Member AIAA.

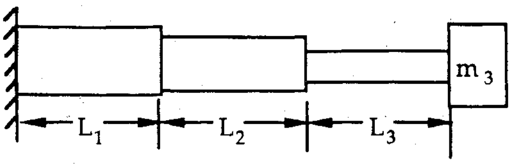


**Fig. 3 Three-degree-of-freedom system with a tip mass.**

$$[M] = [M_s(x)] + [M_{ns}] \tag{3}$$

where $[M_{ns}]$ is the contribution to the mass matrix due to nonstructural mass.

Note that if we ignore this structural mass, the mass matrix is independent of the design variables. Both $[K]$ and $[M]$ are tridiagonal matrices. The structural mass matrix $[M_s(x)]$ depends on whether a lumped mass or a consistent mass model is used.

## Main Results

The constrained optimization problem as stated in problem A can be solved by the Lagrange multiplier technique. Form the Lagrangian as $L = W - \mu(\lambda_1 - p)$. The optimality criteria are

$$\frac{\partial L}{\partial x_i} = \frac{\partial W}{\partial x_i} - \mu \frac{\partial \lambda}{\partial x_i} = 0 \tag{4}$$

where

$$\frac{\partial W}{\partial x_i} = c_i \tag{5}$$

For the system shown in Fig. 1, the eigenvalue derivative can be derived as[7]

$$\frac{\partial \lambda_1}{\delta x_i} = \frac{(\phi_i - \phi_{i-1})^2}{M_1} \tag{6}$$

where $M_i$ is the generalized mass of the first mode, $\phi_i$ the mode shape coefficient of the first mode at degree of freedom $i$, and $\phi_0 = 0$. Substituting Eqs. (5) and (6) into Eq. (4), the optimality criteria become

$$M_1/\mu = \text{const} = (\phi_i - \phi_{i-1})^2/c_i \tag{7}$$

If we define $\phi_1 = \sqrt{c_1}$ and note that $\phi_j = \phi_i$ for $j > i$, we have proven the following theorem.

#### Theorem 1: Mode for Optimal Design

The mode shape corresponding to the optimal design of problem A is given, for $i = 1$ to $n$, by

$$\phi_i = \sum_{j=1}^{i} \sqrt{c_j} \tag{8}$$

Once we know the optimal mode shape, the optimal design can be computed by substituting the optimal mode shape and $p$ for $\lambda_1$ in the eigenvalue problem, Eq. (1). The results are summarized in the following.

#### Theorem 2: Optimal Design Formula for Problem A

The optimal spring constants for problem A are given, for $i = 1$ to $n$, by

$$x_i^* = \frac{p}{\sqrt{c_i}} \left[ \sum_{j=1}^{n} m_j \phi_j \right] \tag{9}$$

where $m_j$ is the lumped mass and $\phi_j$ the optimal mode shape coefficient given by Eq. (8).

## Effects of Structural Mass

When the contribution to mass matrix due to structural elements is considered, the results are more involved. Our closed form solution becomes recursive. Following similar procedures as those used in deriving Eq. (8), we can prove the following theorem.

#### Theorem 3: Optimal Mode Shape for Problem B

When structural mass is considered, the optimal mode shape for problem B is given by $\phi_1 = 1$ and for $i = 1$ to $n - 1$,

$$\phi_{i+1} = [1/(2D_{i+1})][-G \pm \sqrt{G^2 - 4D_{i+1}H}] \tag{10}$$

where

$$G = F_{i+1}\phi_i \tag{11}$$

$$H = D_{i+1}\phi_i^2 - [(c_{i+1})/c_i](D_i\phi_i^2 + F_i\phi_{i-1}\phi_i + D_i\phi_{i-1}^2) \tag{12}$$

$$D_i = 1 - p\alpha_i \tag{13}$$

$$F_i = -2(1 + p\beta_i) \tag{14}$$

For lumped mass model, $\alpha_i = 0.5s_i$, $\beta_i = 0$; for consistent mass model $\alpha_i = s_i/3$, $\beta_i = s_i/6$, and $s_i = (\rho_i L_i^2)/E_i$. Note that Eq. (10) yields two solutions for $\phi_{i+1}$; we should choose $\phi_{i+1} > \phi_i$. We need to use $\phi_0 = 0$, and, when computing $\phi_2$, we use Eq. (10).

With the optimal mode shape known, we can follow the same procedure of deriving Eq. (9) to derive formulas for optimal design. The results are summarized in the following theorem.

#### Theorem 4: Optimal Design Formula for Problem B

The optimal spring constants for problem B can be computed recursively for $i = n$ to 1:

$$x_i^* = \frac{x_{i+1}^*[(\phi_{i+1} - \phi_i) + p(\alpha_{i+1}\phi_i + \beta_{i+1}\phi_{i+1}) + m_i p \phi_i]}{\phi_i(1 - p\alpha_i) - \phi_{i-1}(1 + p\beta_i)} \tag{15}$$

Note that we use $x_{n+1}^* = 0$ when computing $x_n^*$. The optimal cross-sectional areas can be computed from

$$A_i^* = (L_i/E_i)x_i^* \tag{16}$$

## Numerical Example

Consider the three-degrees-of-freedom system of Fig. 3. The minimum mass design with a specified natural frequency was reported by Turner.[1] In this model, each structural element has a length of 30 in.; they are all made of a material with Young's modulus $10.3 \times 10^6$ psi and mass density $2.59 \times 10^{-4}$ lb-s$^2$/in.[4] The desired first mode frequency is 410 Hz. Thye system carries a tip mass of 0.1 lb-s$^2$/in. Using this data, we compute the optimal mode shape as $\{\phi_1\} = [1\ 2.16\ 3.66]^T$, which can be normalized to $\{\phi_1\} = [0.273\ 0.590\ 1.000]^T$. This is very similar to Turner's results.

Using the results of Theorem 4, the optimal spring constants are computed to be $\{X^*\}^T = [4234798\ 3187170\ 1922990]$ lb/in., and the corresponding optimal cross-sectional areas are $\{A^*\} = [12.33\ 9.28\ 5.60]$ (in.$^2$), which is very close to Turner's results. Note that consistent mass matrix was used in the preceding calculations.

## References

[1]Turner, M. J., "Design of Minimum Mass Structures with Specified Natural Frequencies," *AIAA Journal*, Vol. 5, No. 3, 1967, pp. 406–412.

[2]Kiusalaa, J., and Shaw, R. C. J., "An Algorithm for Optimal Structural Design and Frequency Constraints," *International Journal for Numerical Methods in Engineering*, Vol. 13, No. 2, 1978, pp. 283–295.

[3]Khot, N. S., "Optimization of Structures with Multiple Frequency Constraints," *Computers and Structures*, Vol. 20, No. 5, 1985, pp. 869–876.

[4]Pederson, P., "The Integrated Approach of FEM-SLP for Solving Problems of Optimal Design," *Optimization of Distributed Parameter Structures*, edited by J. Cea, and E. J. Haug, Sijthoff and Nourdhoff, Rockville, MD, 1981, pp. 757–780.

[5]Woo, T. H., "Space Frame Optimization Subject to Frequency Constraints," *Proceedings of AIAA/ASME/ASCE/AHS 27th Structures, Structural Dynamics and Materials Conferences*, Vol. 1, AIAA, New York, 1986, pp. 103–115.

[6]Wang, B. P., "Synthesis of Structures with Multiple Frequency Constraints," *Proceedings of AIAA/ASME/ASCE/AHS 27th Structures, Structural Dynamics and Materials Conference*, Vol. 1, 1986, pp. 394–397.

[7]Fox, R. L., and Kapoor, M. P., "Rate of Change of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, No. 12, 1968, pp. 337–351.